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Coherent state quantization of a particle in de Sitter space

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Abstract

We present a coherent state quantization of the dynamics of a relativistic test particle on a one-sheet hyperboloid embedded in a three-dimensional Minkowski space. The group $SO_0(1, 2)$ is considered to be the symmetry group of the system. Our procedure relies on the choice of coherent states of the motion on a circle. The coherent state realization of the principal series representation of $SO_0(1, 2)$ seems to be a new result.

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1. Introduction

In this paper we carry out a coherent state quantization of the dynamics of a relativistic test particle on a one-sheet hyperboloid embedded in a three-dimensional Minkowski space, i.e., on the two-dimensional de Sitter space with the topology $\mathbb{R} \times \mathbb{S}$.

Quantizing the dynamics of a free particle in a curved spacetime is not a purely academic issue since it occurs in fundamental problems. For instance, there is a great interest in cosmology owing to the puzzle of dark matter and energy (see [1, 2] and references therein). Models constructed to solve it within the framework of higher dimensional theories assume that baryonic matter occurs only on one brane embedded in a higher dimensional space (see, e.g., [3–6]). It is important in the context of brane cosmology to understand the confinement problem of a particle to the brane. Examination of the classical particle confinement [7, 8] is not sufficient because elementary particles are quantum objects. Quantization of particle dynamics on a two-dimensional hyperboloid embedded in a three-dimensional Minkowski space may be treated to some extent as a toy model of this problem. Another example is quantization of particle dynamics in singular spacetimes in order to see what one can do

to avoid problems connected with singularities in quantum theory. The case of removable-type singularities has been considered in [9, 10]. It is important to extend these analyses to spacetimes with essential-type singularities because the results may bring some new ideas useful in the construction of quantum gravity.

The problem of a particle in de Sitter space has already been solved rigorously within the group theory oriented quantization scheme [10, 11]. Since the coherent state method seems to be powerful but is still under development, it makes sense to compare its effectiveness with the group theoretical one. That is the goal of this paper. In what follows, we use the results of [10, 11] concerning the classical dynamics of a particle. The quantization method described in [12, 13] is applied to find the corresponding quantum dynamics. More specifically, we test a method based on adapted coherent states and inspired by the Berezin approach [14–18]. It seems to be applicable to situations when the canonical quantization method fails owing to, e.g., the operator ordering (see [19] and references therein) and irreducibility of representation problems [20].

Our paper is organized as follows: in section 2, we specify the canonical structure of our system which is suitable for quantization by the ‘first reduce and then quantize’ method. The choice of coherent states of a particle on a hyperboloid, based on the choice of coherent states of a particle on a circle, is presented in section 3. Both the coherent states and quantum observables are parametrized by some real parameters $\epsilon > 0$. We show in section 4 that in the limit $\epsilon \rightarrow 0$ our coherent state method leads to the Lie algebra homomorphism of classical observables into quantum observables. In section 5, we relate our representation to the Bargmann principal series representation of $SU(1, 1)$ group. We shortly discuss our results in section 6. The two appendices should help the reader to follow easily some technical aspects of the paper.

2. Phase space and observables

In the context of this paper, the coherent state quantization will play the role of a canonical quantization. Thus, it needs the specification of canonical phase space, observables and symmetries of the system. To make the paper self-contained, we recall the main steps of our paper [10] concerning the classical dynamics.

The two-dimensional de Sitter space \mathbb{V} with the topology $\mathbb{R} \times \mathbb{S}$ may be visualized as a one-sheet hyperboloid \mathbb{H}_{r_0} embedded in a three-dimensional Minkowski space \mathbb{M} , i.e.

$$\mathbb{H}_{r_0} := \{(y^0, y^1, y^2) \in \mathbb{M} \mid (y^2)^2 + (y^1)^2 - (y^0)^2 = r_0^2, r_0 > 0\} \quad (1)$$

where r_0 is the parameter of the one-sheet hyperboloid \mathbb{H}_{r_0} .

It is commonly known that the induced metric, $g_{\mu\nu}$ ($\mu, \nu = 0, 1$), on \mathbb{H}_{r_0} is the de Sitter metric.

An action integral, \mathcal{A} , describing a free relativistic particle of mass $m_0 > 0$ in gravitational field $g_{\mu\nu}$ is proportional to the length of a particle world-line and is given by

$$\mathcal{A} = \int_{\tau_1}^{\tau_2} L(\tau) d\tau \quad L(\tau) := -m_0 \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \quad (2)$$

where τ is an evolution parameter, x^μ are intrinsic coordinates and $\dot{x}^\mu := dx^\mu/d\tau$. It is assumed that $\dot{x}^0 > 0$, i.e., x^0 has interpretation of time monotonically increasing with τ .

The action (2) is invariant under the reparametrization $\tau \rightarrow f(\tau)$ of the world-line (where f is an arbitrary function of τ). This gauge symmetry leads to the constraint

$$G := g^{\mu\nu} p_\mu p_\nu - m_0^2 = 0 \quad (3)$$

where $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and $p_\alpha := \partial L / \partial \dot{x}^\alpha$ ($\alpha = \mu, \nu$) are canonical momenta.

Since a test particle does not modify spacetime, the local symmetry of spacetime coincides with the algebra of all Killing vector fields. It is known (see, e.g., [21]) that a Killing vector field Y may be used to find a dynamical integral D of a test particle moving along a geodesic by

$$D = p_\mu Y^\mu \quad \mu = 0, 1 \tag{4}$$

where Y^μ are components of Y .

To be more specific we parametrize the hyperboloid (1) as follows [10]:

$$y^0 = -\frac{r_0 \cos \rho/r_0}{\sin \rho/r_0} \quad y^1 = \frac{r_0 \cos \theta/r_0}{\sin \rho/r_0} \quad y^2 = \frac{r_0 \sin \theta/r_0}{\sin \rho/r_0} \tag{5}$$

where $0 < \rho < \pi r_0$ and $0 \leq \theta < 2\pi r_0$.

For the parametrization (5), the metric tensor induced on \mathbb{H}_{r_0} reads

$$ds^2 = (d\rho^2 - d\theta^2) \sin^{-2}(\rho/r_0). \tag{6}$$

Thus, the constraint (3) has the form

$$G = (p_\rho^2 - p_\theta^2) \sin^2(\rho/r_0) - m_0^2 = 0 \tag{7}$$

where $p_\rho := \partial L/\partial \dot{\rho}$ and $p_\theta := \partial L/\partial \dot{\theta}$ are canonical momenta.

The Killing vector fields Y_a ($a = 0, 1, 2$) of our two-dimensional de Sitter space may be found [10] by specification of the infinitesimal transformations of the proper orthochronous Lorentz group $SO_0(1, 2)$. These transformations, in parametrization (5), read

$$(\rho, \theta) \longrightarrow (\rho, \theta + a_0 r_0) \tag{8}$$

$$(\rho, \theta) \longrightarrow (\rho - a_1 r_0 \sin \rho/r_0 \sin \theta/r_0, \theta + a_1 r_0 \cos \rho/r_0 \cos \theta/r_0) \tag{9}$$

$$(\rho, \theta) \longrightarrow (\rho + a_2 r_0 \sin \rho/r_0 \cos \theta/r_0, \theta + a_2 r_0 \cos \rho/r_0 \sin \theta/r_0) \tag{10}$$

where $(a_0, a_1, a_2) \in \mathbb{R}^3$ are small parameters. The transformation (8) corresponds to the infinitesimal rotation interpreted here as space de Sitter translation, whereas (9) and (10) define two infinitesimal boosts. One of them can be interpreted as Lorentz boost, another one describes de Sitter ‘time’ translation.

The dynamical integrals J_a ($a = 0, 1, 2$) defined by (4) and corresponding to (8)–(10), respectively, read

$$J_0 = p_\theta r_0 \tag{11}$$

$$J_1 = -p_\rho r_0 \sin \rho/r_0 \sin \theta/r_0 + p_\theta r_0 \cos \rho/r_0 \cos \theta/r_0 \tag{12}$$

$$J_2 = p_\rho r_0 \sin \rho/r_0 \cos \theta/r_0 + p_\theta r_0 \cos \rho/r_0 \sin \theta/r_0. \tag{13}$$

Making use of (11)–(13), one may rewrite the constraint (7) as

$$J_2^2 + J_1^2 - J_0^2 - \kappa^2 = 0 \quad \kappa := m_0 r_0. \tag{14}$$

One can verify [10] that equations (5) and (11)–(13) lead to the algebraic equations

$$J_a y^a = 0 \quad J_2 y^1 - J_1 y^2 = r_0^2 p_\rho \tag{15}$$

which determine a particle geodesic for given values of J_a ($a = 0, 1, 2$). Equations (15) do not ‘underdetermine’ a particle geodesic because the equation

$$(y^2)^2 + (y^1)^2 - (y^0)^2 = r_0^2 \tag{16}$$

defining the spacetime \mathbb{H}_{r_0} should be satisfied too.

The physical phase space Γ is defined in [10] to be the space of all particle geodesics consistent with the constraint (3). Since each triple (J_0, J_1, J_2) satisfying (14) defines uniquely

a particle geodesic, by solution of (15), the one-sheet hyperboloid (14) represents Γ . For the same reason, it is natural to choose J_a ($a = 0, 1, 2$) to represent the basic observables of our system. One may further argue that we need to make the measurement of the numerical values of all three observables to identify the geodesic of a particle.

It is worthy to compare the present situation with its Minkowskian counterpart. In the latter case, there are also three Killing vector fields and corresponding dynamical integrals: P_0 (time translation), P (space translation) and K (Lorentz boost). Due to the Minkowskian space homogeneity the Lorentz boost is free of constraints so we are left with mass-shell condition which reads $P_0^2 - P^2 - m_0^2 = 0$ and is the Minkowskian counterpart of (14).

The system of a free particle in curved spacetime defined by the action (2) may be treated as a gauge system in which the gauge invariance is the reparametrization invariance. It is a characteristic feature of such a system that the Hamiltonian corresponding to the Lagrangian (2) identically vanishes. The general treatment of such gauge system within the constrained Hamiltonian formalism and the reduction scheme to gauge invariant variables has been presented elsewhere (see [22] and references therein). Here, we only outline the method leading to the phase space with independent canonical variables.

The Hamiltonian formulation of a theory with gauge invariant Lagrangian [23, 24] leads to an extended $2N$ -dimensional phase space Γ_e and M first-class constraints (N denotes spacetime dimension and equals 2 in our case, the constraint is defined by (3) so $M = 1$). The constraint surface Γ_c , defined, e.g., by (7) in Γ_e parametrized by $(\rho, \theta, p_\rho, p_\theta)$, plays special role in the formalism. Our type of system may have up to $2N - 2M$ (that equals 2 in our case) gauge invariant functionally independent variables on Γ_c which may be used to parametrize the physical phase space and gauge invariant observables [23]. It is known [22] that the dynamical integrals may be used to represent such variables. The observables J_0, J_1 and J_2 are gauge invariant (each of them has a vanishing Poisson bracket with G on Γ_c) and any two of them are functionally independent on Γ_c due to (14). However, such two variables do not have the canonical form (i.e., they do not form a conjugate pair) used in a group theoretical quantization scheme. There exists a general method of finding the corresponding canonical variables, but it is quite involved [22]. In what follows, we recall the simple method used in [10]. It consists of three steps.

First, we identify the algebra the basic observables J_a ($a = 0, 1, 2$) satisfy on Γ_c . Direct calculations lead to

$$\{J_0, J_1\} = -J_2 \quad \{J_0, J_2\} = J_1 \quad \{J_1, J_2\} = J_0 \quad (17)$$

which means that our basic observables satisfy $sl(2, \mathbb{R})$ algebra.

Second, we find that the physical phase space Γ , identified with the hyperboloid (14), is diffeomorphic to the manifold X defined to be

$$X := \{x \equiv (J, \beta) \mid J \in \mathbb{R}, 0 \leq \beta < 2\pi\} \quad (18)$$

where the diffeomorphism is given by

$$J_0 := J \quad J_1 := J \cos \beta - \kappa \sin \beta \quad J_2 := J \sin \beta + \kappa \cos \beta \quad 0 < \kappa < \infty. \quad (19)$$

Third, we find that the Poisson bracket on Γ_c in terms of the variables J and β reads

$$\{\cdot, \cdot\} := \frac{\partial \cdot}{\partial J} \frac{\partial \cdot}{\partial \beta} - \frac{\partial \cdot}{\partial \beta} \frac{\partial \cdot}{\partial J}. \quad (20)$$

Since $\{J, \beta\} = 1$, the variables J and β are canonical and X will be called a canonical phase space in what follows. The specification of the canonical structure of our system is now complete.

The coherent state quantization does not require the independent degrees of freedom to form conjugate pairs. But we are going to use the same canonical structure as in the case of

group theoretical quantization to make straightforward the comparison of the results obtained by both methods.

Now, let us identify the symmetry group of the system. The local symmetry is defined by the algebra (17), but it is unclear what the global symmetry could be because there are infinitely many Lie groups having $sl(2, \mathbb{R})$ as their Lie algebras. The common examples are: $SO_0(1, 2)$, the proper orthochronous Lorentz group; $SU(1, 1) \sim SL(2, \mathbb{R})$, the two-fold covering of $SO_0(1, 2) \sim SU(1, 1)/\mathbb{Z}_2$; $\widetilde{SL(2, \mathbb{R})}$, the infinite-fold covering of $SO_0(1, 2) \sim \widetilde{SL(2, \mathbb{R})}/\mathbb{Z}$ (the universal covering group).

Assuming that the symmetry of the system of a particle in de Sitter space is defined by continuous transformations only, it is difficult to identify the symmetry group. However, we are free to use discrete symmetries of our system as well. Since the system of a particle on a hyperboloid is a non-dissipative one, it must be invariant with respect to time-reversal transformations. We have shown in [11] that making use of this symmetry leads to the conclusion that the global symmetry of our system may be either the group $SO_0(1, 2)$ or $SU(1, 1)$.

For the purpose of clarity of presentation of our coherent state results, we carry out further discussion assuming that the symmetry group is $SO_0(1, 2)$. The case of $SU(1, 1)$ symmetry will be considered elsewhere [25].

The problem of quantization of the basic observables (11)–(13) reduces to the problem of finding an (essentially) self-adjoint representation of $sl(2, \mathbb{R})$ algebra integrable to an irreducible unitary representation of $SO_0(1, 2)$ group.

3. Choice of coherent states

In this section, we introduce the quantities which are used in the coherent state quantization [12, 13]. First, we define a measure on X which depends on some non-negative real parameter ϵ as follows:

$$\mu_\epsilon(dx) := \sqrt{\frac{\epsilon}{\pi}} \frac{1}{2\pi} e^{-\epsilon J^2} d\beta dJ. \tag{21}$$

Next, we introduce an abstract separable Hilbert space \mathcal{H} with an orthonormal basis $\{|m\rangle\}_{m \in \mathbb{Z}}$, i.e.,

$$\langle m_1 | m_2 \rangle = \delta_{m_1, m_2} \quad \sum_{m=-\infty}^{+\infty} |m\rangle \langle m| = \mathbb{I}. \tag{22}$$

Then, we construct an orthonormal set of vectors $\{\phi_m^\epsilon\}_{m \in \mathbb{Z}}$ which spans a Hilbert subspace $\mathcal{K}_\epsilon \subset L^2(X, \mu_\epsilon)$, which is peculiar to our physical system.

Being inspired by the choice of the coherent states for the motion of a particle on a circle [26–30], we define ϕ_m^ϵ by

$$\phi_m^\epsilon(J, \beta) := \exp\left(-\frac{\epsilon m^2}{2}\right) \exp(\epsilon m J - im\beta). \tag{23}$$

One may check that

$$\begin{aligned} \mathcal{N}_\epsilon(x) \equiv \mathcal{N}_\epsilon(J, \beta) &:= \sum_{m=-\infty}^{+\infty} |\phi_m^\epsilon|^2 = \sum_{m=-\infty}^{+\infty} \exp(\epsilon(2mJ - m^2)) \\ &= \vartheta_3(i\epsilon J, e^{-\epsilon}) < \infty \end{aligned} \tag{24}$$

where the elliptic theta function [31] is given by

$$\vartheta_3(z, q) = \sum_{m=-\infty}^{+\infty} q^{m^2} e^{2miz}.$$

Finally, we define the coherent states $|x, \epsilon\rangle$ as follows:

$$X \ni x \longrightarrow |x, \epsilon\rangle \equiv |J, \beta, \epsilon\rangle := \frac{1}{\sqrt{\mathcal{N}_\epsilon(J, \beta)}} \sum_{m=-\infty}^{+\infty} \phi_m^\epsilon(J, \beta) |m\rangle \in \mathcal{H}_\epsilon \quad (25)$$

where \mathcal{H}_ϵ is an abstract Hilbert space spanned by $\{|x, \epsilon\rangle\}_{x \in X}$.

The states defined by (25) are easily verified to be *coherent* in the sense that they satisfy the normalization condition

$$\langle x, \epsilon | x, \epsilon \rangle = \frac{1}{\mathcal{N}_\epsilon} \sum_{m_1, m_2} \langle m_1 | \overline{\phi_{m_1}^\epsilon} \phi_{m_2}^\epsilon | m_2 \rangle = 1 \quad (26)$$

and lead to the resolution of the identity in \mathcal{H}_ϵ

$$\int_X \mu_\epsilon(dx) \mathcal{N}_\epsilon(x) |x, \epsilon\rangle \langle x, \epsilon| = \sqrt{\frac{\epsilon}{\pi}} \frac{1}{2\pi} \int_0^{2\pi} d\beta \int_{-\infty}^{+\infty} dJ \mathcal{N}_\epsilon(J, \beta) e^{-\epsilon J^2} |J, \beta, \epsilon\rangle \langle J, \beta, \epsilon| = \mathbb{I}. \quad (27)$$

Localization properties of our coherent states are well illustrated by the behaviour of their respective ‘overlaps’ $\langle J', \beta', \epsilon | J, \beta, \epsilon \rangle$ as functions of (J', β')

$$\langle J', \beta', \epsilon | J, \beta, \epsilon \rangle \equiv \Psi_{J, \beta}^\epsilon(J', \beta') = \frac{\vartheta_3(i\epsilon \frac{J+J'}{2} + \frac{\beta-\beta'}{2}, e^{-\epsilon})}{\sqrt{\vartheta_3(i\epsilon J, e^{-\epsilon}) \vartheta_3(i\epsilon J', e^{-\epsilon})}}. \quad (28)$$

A comprehensive discussion of this issue will be presented elsewhere [25].

4. Homomorphism

In the coherent state quantization method [12, 13], a quantum operator is defined by the mapping

$$f \longrightarrow \hat{f}^\epsilon := A_\epsilon(f) := \int_X \mu_\epsilon(dx) \mathcal{N}_\epsilon(x) f(x) |x, \epsilon\rangle \langle x, \epsilon| \quad (29)$$

where $f = f(J, \beta)$ is a classical observable, and where the integral is interpreted in the weak sense whenever it exists. In general, an arbitrary unbounded operator on the Hilbert space \mathcal{H}_ϵ may or may not possess a pseudo-diagonal representation like (29). The domain of the operator \hat{f}^ϵ is defined to be a dense subspace of \mathcal{H}_ϵ . It is clear that an explicit form of this domain depends on the properties of the function f (integrability, C^∞ , etc).

Making use of the formulae of appendix A, we obtain

$$\hat{J}_0^\epsilon = A_\epsilon(J_0) = \sum_{m=-\infty}^{+\infty} m |m\rangle \langle m| \quad (30)$$

$$\hat{J}_1^\epsilon = A_\epsilon(J_1) = \frac{1}{2} e^{-\epsilon/4} \sum_{m=-\infty}^{+\infty} \left(\left(m + \frac{1}{2} + i\kappa \right) |m+1\rangle \langle m| + \text{c.c.} \right) \quad (31)$$

$$\hat{J}_2^\epsilon = A_\epsilon(J_2) = \frac{1}{2i} e^{-\epsilon/4} \sum_{m=-\infty}^{+\infty} \left(\left(m + \frac{1}{2} + i\kappa \right) |m+1\rangle \langle m| - \text{c.c.} \right) \quad (32)$$

where c.c. stands for the complex conjugate of the preceding term.

The commutation relations corresponding to (17) are found to be

$$[\hat{J}_0^\epsilon, \hat{J}_1^\epsilon] = i\hat{J}_2^\epsilon = -iA_\epsilon(\{J_0, J_1\}) \tag{33}$$

$$[\hat{J}_0^\epsilon, \hat{J}_2^\epsilon] = -i\hat{J}_1^\epsilon = -iA_\epsilon(\{J_0, J_2\}) \tag{34}$$

$$[\hat{J}_1^\epsilon, \hat{J}_2^\epsilon] = -ie^{-\epsilon/2}\hat{J}_0^\epsilon = e^{-\epsilon/2}(-iA_\epsilon(\{J_1, J_2\})). \tag{35}$$

Now, we consider the asymptotic case $\epsilon \rightarrow 0$. All operators and equations in this limit are defined in the abstract Hilbert space \mathcal{H} . The equations (33)–(35) prove that in the asymptotic case the mapping (29) is a homomorphism.

5. Representation

We prove in appendix B that our representation of $sl(2, \mathbb{R})$ algebra, at the limit $\epsilon \rightarrow 0$, is essentially self-adjoint.

The classification of all irreducible unitary representations of $SO_0(1, 2)$ group has been done by Bargmann [32]. To identify our representation within Bargmann’s classification, we consider the Casimir operator. The classical Casimir operator C for the algebra (17) has the form

$$C = J_2^2 + J_1^2 - J_0^2 = \kappa^2. \tag{36}$$

Making use of (29), we map C into the corresponding quantum operator \hat{C} as follows:

$$\begin{aligned} \hat{C} &= \lim_{\epsilon \rightarrow 0} \hat{C}^\epsilon := \lim_{\epsilon \rightarrow 0} (\hat{J}_2^\epsilon \hat{J}_2^\epsilon + \hat{J}_1^\epsilon \hat{J}_1^\epsilon - \hat{J}_0^\epsilon \hat{J}_0^\epsilon) \\ &= \lim_{\epsilon \rightarrow 0} \sum_{m=-\infty}^{+\infty} \left(e^{-\epsilon/2} \left(m^2 + \kappa^2 + \frac{1}{4} \right) - m^2 \right) |m\rangle\langle m| \\ &= \left(\kappa^2 + \frac{1}{4} \right) \sum_{m=-\infty}^{+\infty} |m\rangle\langle m| = \left(\kappa^2 + \frac{1}{4} \right) \mathbb{I} =: q\mathbb{I}. \end{aligned} \tag{37}$$

Thus, we have obtained that our choice of coherent states (25) and the mapping (29) lead, as $\epsilon \rightarrow 0$, to the representation of $sl(2, \mathbb{R})$ algebra with the Casimir operator to be an identity in \mathcal{H} multiplied by a real constant $1/4 < q < \infty$.

In the asymptotic case, $\hat{J}_a := \lim_{\epsilon \rightarrow 0} J_a^\epsilon$ ($a = 0, 1, 2$), all operators are defined in the Hilbert space \mathcal{H} . The specific realization of \mathcal{H} may be obtained by finding eigenfunctions of the set of all commuting observables of the system. It is easy to verify that for our system the commuting observables are \hat{C} and \hat{J}_0 . The set of common eigenfunctions may be chosen to be

$$\phi_m(J, \beta) = \exp(im\beta) \quad m \in \mathbb{Z} \tag{38}$$

which spans $L^2(\mathbb{S}^1, \mu)$ with $\mu(dx) := d\beta/2\pi$.

The Bargmann classification is characterized by the ranges of q and m . The range of our parameter κ is $0 < \kappa < \infty$, so it corresponds to Bargmann’s $1/4 < q < \infty$. Since $m \in \mathbb{Z}$ in our choice of \mathcal{H} , the coherent state representation (in an asymptotic sense) we have found is almost everywhere identical to the Bargmann’s continuous integral case irreducible unitary representation called C_q^0 , with $1/4 \leq q < \infty$, and this corresponds to the principal series representation of $SO_0(1, 2)$ group [33]. The only difference is that for massive particle we have $m_0 > 0$, thus $\kappa = m_0 r_0 > 0$, so $q > 1/4$. Our representation coincides with C_q^0 representation in case taking the limit $\kappa \rightarrow 0$, i.e., $m_0 \rightarrow 0$, makes sense. We discuss this issue in our next paper [25] in relation with the work [34].

6. Discussion

As is mentioned in section 1, in the group theoretical quantization method there is a problem with the ordering of canonical operators appearing in the process of mapping classical observables into the corresponding quantum operators. One does not know how to solve this problem in the case in which observables are higher than first-order polynomials in one half of the canonical coordinates and related no-go theorems (see [35] for a recent review on polynomial quantization problems). In the case of particle dynamics on hyperboloid, we have managed to find such two canonical variables that the basic observables (19) are linear functions in one of them [10], but our method cannot be generalized to any observables.

On the other hand, there is no problem with the ordering of operators in the coherent state quantization method presented here. Indeed, the mapping (29) includes only classical expression for observables. This is the main feature of the method, since (29) may be applied to almost any classical observable, i.e., smooth function of canonical variables, and even to more exotic functions. The key ingredient of the method is the definition of the coherent states, in particular the construction of the set of orthonormal vectors (23) which span the Hilbert space \mathcal{K}_ϵ . We have solved this problem due to the existence of a certain family of coherent states associated with the quantum motion of a particle on the unit circle. It is clear that new physical problems call for new constructions of \mathcal{K}_ϵ .

As far as we know our paper presents the construction of the principal series representation of $SO_0(1, 2)$ group by making use of the appropriate coherent state quantization for the first time. Let us recall that the representations of the discrete series have already been obtained within a coherent state framework [14, 15]. It would be interesting to extend these results to include the complementary series representations as well.

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Appendix A

Application of the mapping (29) to the observables J_a ($a = 0, 1, 2$) reduces to the problem of calculating quantities such as $A_\epsilon(J)$, $A_\epsilon(e^{\pm i\beta})$ and $A_\epsilon(J e^{\pm i\beta})$.

Making use of

$$\exp(\pm i\beta) = \cos \beta \pm i \sin \beta \quad \int_{-\infty}^{+\infty} \exp(-bt^2) dt = \sqrt{\pi/b} \quad b > 0 \quad (\text{A.1})$$

and equation (29), one can easily obtain the following expressions:

$$A_\epsilon(J) = \sum_{m=-\infty}^{+\infty} m |m\rangle \langle m| \quad A_\epsilon(e^{i\beta}) = e^{-\epsilon/4} \sum_{m=-\infty}^{+\infty} |m+1\rangle \langle m| \quad (\text{A.2})$$

$$A_\epsilon(J e^{i\beta}) = e^{-\epsilon/4} \sum_{m=-\infty}^{+\infty} (m+1) |m+1\rangle \langle m|. \quad (\text{A.3})$$

The formulae for $A_\epsilon(e^{-i\beta})$ and $A_\epsilon(J e^{-i\beta})$ are obtained by taking the complex conjugate of $A_\epsilon(e^{i\beta})$ and $A_\epsilon(J e^{i\beta})$, respectively.

Appendix B

To show that the operators \hat{J}_a ($a = 0, 1, 2$) are essentially self-adjoint in \mathcal{H} , we make use of theorem 5 of the Nelson method [36].

The Nelson operator, Δ , for $sl(2, \mathbb{R})$ algebra reads

$$\Delta := \hat{J}_0^2 + \hat{J}_1^2 + \hat{J}_2^2 = \sum_{m=-\infty}^{+\infty} (2m^2 + \kappa^2 + 1/4) |m\rangle \langle m| = \kappa^2 + 1/4 + 2 \sum_{m=-\infty}^{+\infty} m^2 |m\rangle \langle m|. \quad (\text{B.1})$$

It is clear that Δ is unbounded and symmetric in \mathcal{H} . It is also essentially self-adjoint because it is a real diagonal operator.

Since the operator Δ is essentially self-adjoint, it results from the first part of Nelson's theorem that our representation of $sl(2, \mathbb{R})$ algebra is integrable to a unique unitary representation U of the universal covering group $\widetilde{SL}(2, \mathbb{R})$.

The second part of Nelson's theorem states that the application of the Stone theorem [37] to the one-parameter unitary subgroups of U generated by J_a ($a = 0, 1, 2$) gives the essentially self-adjoint operators \hat{J}_a ($a = 0, 1, 2$).

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